Combined Mathematics - I
Third term test – Grade 13 – 2026
North Western Province

01. By using the principle of mathematical induction, prove that $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$ for all $n \in Z^+$.

For $n = 1$, $L.H.S = \frac{1}{2^1} = \frac{1}{2}$ and $R.H.S. = 2 - \frac{1+2}{2^1} = \frac{1}{2}$. $\therefore$ The result is true for $n = 1$. 5

Let the result is true for $n = p; p \in Z^+$. Then $\sum_{r=1}^{p} \frac{r}{2^r} = 2 - \frac{p+2}{2^p}$. 5

Let $n = p+1$. Then $\sum_{r=1}^{p+1} \frac{r}{2^r} = \sum_{r=1}^{p} \frac{r}{2^r} + \frac{p+1}{2^{p+1}} = 2 - \frac{p+2}{2^p} + \frac{p+1}{2^{p+1}}$ 5

$$= 2 - \left(\frac{p+2}{2^p} - \frac{p+1}{2^{p+1}}\right)$$

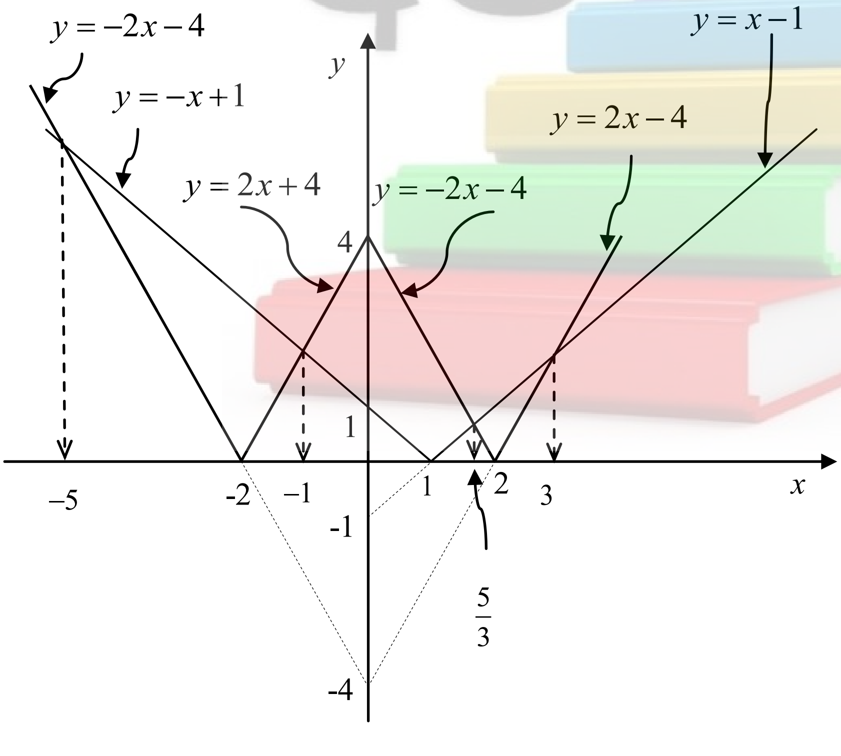
$$= 2 - \frac{2p+4-p-1}{2^{p+1}}$$ 5

$$= 2 - \frac{p+3}{2^{p+1}}$$

$$= 2 - \frac{(p+1)+2}{2^{p+1}}$$

The result is true for $n = p+1$ when it is true for $n = p$. Therefore according to the principle of mathematical induction, the result is true for any positive integer n. 5

02. Sketch the graph of $y = |4 - 2|x||$ and $y = |x-1|$ in the same diagram. Hence or otherwise, find all real values of x satisfying the inequality $|4-2|x|| \geq |x-1|$.



$-2x - 4 = -x + 1$

$\underline{\underline{x = -5}}$

$2x + 4 = -x + 1$

$\underline{\underline{x = -1}}$

$-2x + 4 = x - 1$

$\underline{\underline{x = \frac{5}{3}}}$

$2x - 4 = x - 1$

$\underline{\underline{x = 3}}$

$\therefore$ $|4-2|x|| \geq |x-1|$ for $x \in (-\infty, -5] \cup \left[-1, \frac{5}{3}\right] \cup [3, \infty)$

03. The complex number z satisfies all the three relationships $|z-1| \leq 1$, $\arg(z+1) \geq \frac{\pi}{12}$ and $z+\bar{z} \geq 2$. Shade the region which z is satisfied in an Argand diagram.

$|z-1| \leq 1$

$|z-(1+0i)| \leq 1$

$\arg(z+1) \geq \frac{\pi}{12}$

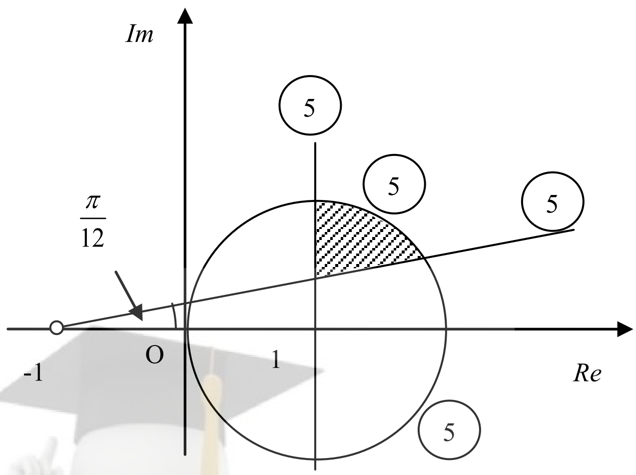
$\arg[z-(-1+0i)] \geq \frac{\pi}{12}$

$z+\bar{z} \geq 2$

$2\mathrm{Re}(z) \geq 2$

$\mathrm{Re}(z) \geq 1$

5



04. The first two terms, in ascending powers of x, in the expansion of $(1+x)\left(2-\frac{x}{4}\right)^n$ are $p+qx^2$. Find the values of n, p and q.

$$(1+x)\left(2-\frac{x}{4}\right)^n = (1+x)\left({}^nC_0 2^n - {}^nC_1 2^{n-1}\left(\frac{x}{4}\right)^1 + {}^nC_2 2^{n-2}\left(\frac{x}{4}\right)^2 - \ldots\right)$$

$$\therefore \quad p+qx^2 = {}^nC_0 2^n + \left({}^nC_0 2^n - {}^nC_1 2^{n-1}\frac{1}{4}\right)x + \left({}^nC_2 2^{n-2}\frac{1}{16} - {}^nC_1 2^{n-1}\frac{1}{4}\right)x^2$$ 10

comparing coefficients of x

$0 = {}^nC_0 2^n - {}^nC_1 2^{n-1}\frac{1}{4}$

$0 = 8\,{}^nC_0 2^{n-3} - {}^nC_1 2^{n-3}$

$0 = 8 \times 2^{n-3} - n2^{n-3}$

$0 = (8-n)2^{n-3}$

Since $2^{n-3} \neq 0$, $n = \underline{\underline{8}}$

5

comparing constants

$p = {}^nC_0 2^n$

$p = {}^8C_0 2^8$

$p = \underline{\underline{256}}$

5

comparing coefficients of x^2

$q = {}^nC_2 2^{n-2}\frac{1}{16} - {}^nC_1 2^{n-1}\frac{1}{4}$

$q = {}^8C_2 2^{8-2}\frac{1}{16} - {}^8C_1 2^{8-1}\frac{1}{4}$

$q = 28 \times 4 - 8 \times 32$

$q = 16(7-16)$

$q = -16 \times 9$

$q = \underline{\underline{-144}}$

5

05. Show that $\lim_{x\to\pi} \frac{\left(1+\cos x\sqrt{\cos 2x}\right)\left(\sin x+\tan x\right)}{\left(x-\pi\right)^5}=\frac{3}{4}$.

$$\lim_{x\to\pi} \frac{\left(1+\cos x\sqrt{\cos 2x}\right)\left(\sin x+\tan x\right)}{\left(x-\pi\right)^5}$$

Let $x-\pi=\theta$. Then $x=\pi+\theta$.

$$\therefore \quad \lim_{x\to\pi} \frac{\left(1+\cos x\sqrt{\cos 2x}\right)\left(\sin x+\tan x\right)}{\left(x-\pi\right)^5}$$

$$=\lim_{\theta\to 0} \frac{\left[1+\cos(\pi+\theta)\sqrt{\cos 2(\pi+\theta)}\right]\left[\sin(\pi+\theta)+\tan(\pi+\theta)\right]}{\theta^5} \quad 5$$

$$=\lim_{\theta\to 0} \frac{\left[1-\cos\theta\sqrt{\cos 2\theta}\right]\left[-\sin\theta+\tan\theta\right]}{\theta^5} \quad 5$$

$$=\lim_{\theta\to 0} \frac{\left[1-\cos^2\theta\cos 2\theta\right]\left[-\cos\theta\sin\theta+\sin\theta\right]}{\theta^5\left[1+\cos\theta\sqrt{\cos 2\theta}\right]\cos\theta}$$

$$=\lim_{\theta\to 0} \frac{\left[1-\cos^2\theta\left(1-2\sin^2\theta\right)\right]\sin\theta\left(1-\cos\theta\right)}{\theta^5\left[1+\cos\theta\sqrt{\cos 2\theta}\right]\cos\theta}$$

$$=\lim_{\theta\to 0} \frac{\left(1-\cos^2\theta+2\sin^2\theta\cos^2\theta\right)\sin\theta\left(1-\cos^2\theta\right)}{\theta^5\left[1+\cos\theta\sqrt{\cos 2\theta}\right]\cos\theta\left(1+\cos\theta\right)}$$

$$=\lim_{\theta\to 0} \frac{\left(\sin^2\theta+2\sin^2\theta\cos^2\theta\right)\sin\theta\sin^2\theta}{\theta^5\left[1+\cos\theta\sqrt{\cos 2\theta}\right]\cos\theta\left(1+\cos\theta\right)} \quad 5$$

$$=\lim_{\theta\to 0} \frac{\sin^2\theta\left(1+2\cos^2\theta\right)\sin^3\theta}{\theta^5\left[1+\cos\theta\sqrt{\cos 2\theta}\right]\cos\theta\left(1+\cos\theta\right)}$$

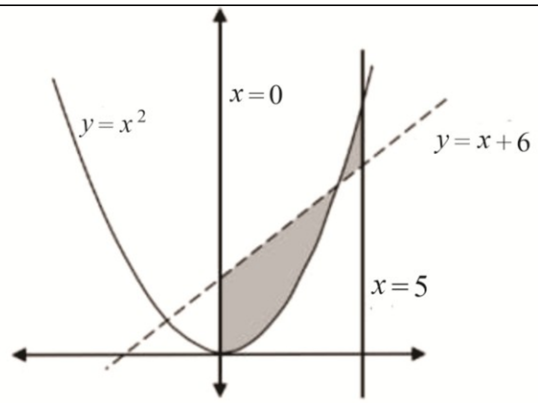
$$=\lim_{\theta\to 0} \frac{\sin^5\theta}{\theta^5}\frac{\left(1+2\cos^2\theta\right)}{\left[1+\cos\theta\sqrt{\cos 2\theta}\right]\cos\theta\left(1+\cos\theta\right)}$$

$$=\lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right)^5 \frac{\left(1+2\cos^2\theta\right)}{\cos\theta\left(1+\cos\theta\right)\left[1+\cos\theta\sqrt{\cos 2\theta}\right]}$$

$$=1^5\times\frac{(1+2\times 1)}{1(1+1)\left[1+1\times\sqrt{1}\right]} \quad 5$$

$$=\underline{\underline{\frac{3}{4}}}$$

06. Find the area of the region enclosed by the curves $y = x^2$, $y = x+6$, $x = 5$ and the y – axis.



$x^2 = x+6$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3 \quad or \quad x = -2$

for given region

$x = 3$ 5

Area of the shaded region

$$= \int_0^3 \left[(x+6) - (x^2) \right] dx + \int_3^5 \left[(x^2) - (x+6) \right] dx \quad 5$$

$$= \int_0^3 \left(x+6-x^2 \right) dx - \int_3^5 \left(x+6-x^2 \right) dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_0^3 - \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_3^5 \quad 5$$

$$= \left(\frac{3^2}{2} + 6\times3 - \frac{3^3}{3} \right) - \left(\frac{5^2}{2} + 6\times5 - \frac{5^3}{3} \right) + \left(\frac{3^2}{2} + 6\times3 - \frac{3^3}{3} \right) \quad 5$$

$$= \frac{18-25}{2} + 36 - 30 - \frac{54-125}{3}$$

$$= \frac{-7}{2} + 6 + \frac{71}{3}$$

$$= \frac{-21+36+142}{6}$$

$$= \frac{157}{6} \quad \textit{square units} \quad 5$$

07. The parametric equations of a curve are given by $x = 4t-1$, $y = \dfrac{5}{2t} + 10$; $t \in R$, $t \neq 0$, where t is a parameter. The curve crosses the x - axis at the point A. Find the coordinates of A and the equation of the tangent drawn to the curve at the point A.

$$x = 4t-1 \;\Rightarrow\; \frac{dx}{dt} = 4, \qquad y = \frac{5}{2t} + 10 \;\Rightarrow\; \frac{dy}{dt} = \frac{5}{2}(-1)x^{-2} = -\frac{5}{2t^2} \quad 5$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -\frac{5}{2t^2} \cdot \frac{1}{4} = -\frac{5}{8t^2}$$

Since the point A is on the x - axis, let $A \equiv (x_A, 0)$ 5

$$\therefore \quad 0 = \frac{5}{2t} + 10 \;\Rightarrow\; 20t = -5 \;\Rightarrow\; t = -\frac{1}{4} \quad 5$$

$\therefore \quad x_A = 4\times\left(-\frac{1}{4}\right)-1=-2, \therefore \quad A\equiv(-2,0)$

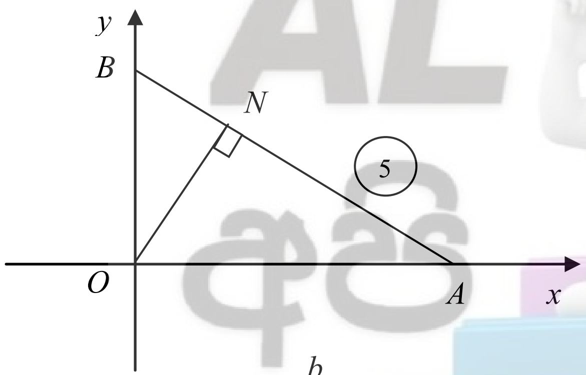
The gradient of the tangent at A, $m_A = \left(\frac{dy}{dx}\right)_{t=-\frac{1}{4}} = -\frac{5}{8\times\frac{1}{16}} = -10$ 5

Equation of the tangent at A is given by

$\frac{y-0}{x+2} = -10 \quad \Rightarrow \quad y=-10x-20$. Equation of the tangent is $\underline{\underline{10x+y+20=0}}$

$10x+y+20=0$ 5

08. A and B are two variable points on the positive directions of x - axis and y - axis respectively, such that $OA+OB=\mu$. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x+y)(x^2+y^2)=\mu xy$.



Let $A\equiv(a,0)$ and $B\equiv(0,b)$.
Also let the foot of perpendicular is N and $N\equiv(\overline{x},\overline{y})$.

Given that $OA+OB=\mu$.

$\therefore \quad a+b=\mu \quad \rightarrow \quad (2)$ 5

Gradient of $AB = -\frac{b}{a}$ and gradient of $ON = \frac{\overline{y}}{\overline{x}}$.

Since AB and ON are perpendicular

$\left(-\frac{b}{a}\right)\frac{\overline{y}}{\overline{x}}=-1 \quad \Rightarrow \quad b\overline{y}=a\overline{x} \quad \rightarrow (1)$ 5

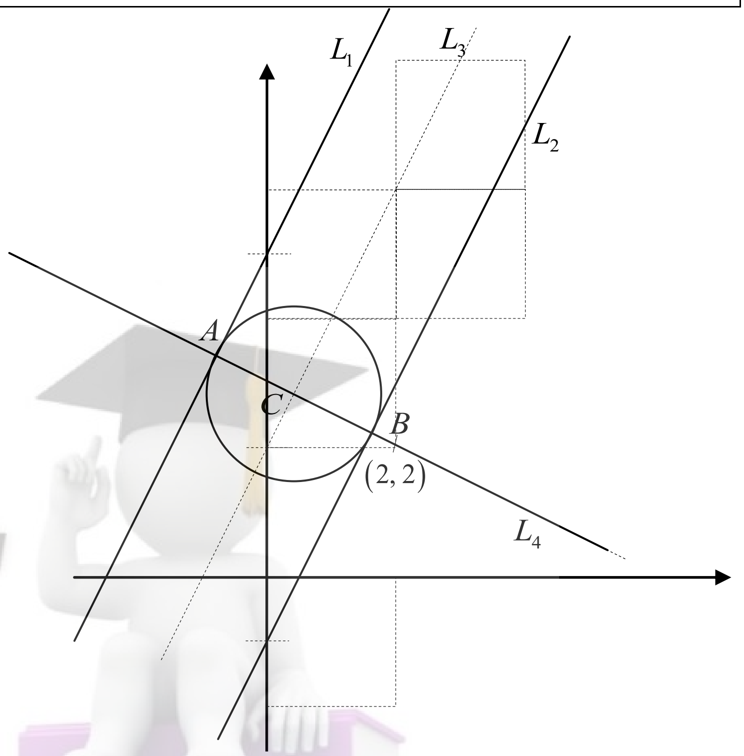
$(1),(2) \quad \Rightarrow \quad b=\frac{\mu\overline{x}}{\overline{x}+\overline{y}}, \quad a=\frac{\mu\overline{y}}{\overline{x}+\overline{y}}$.

$\frac{\overline{y}-0}{\overline{x}-a}=-\frac{b}{a} \quad \Rightarrow \quad a\overline{y}+b\overline{x}=ab$ 5

$\frac{\mu\overline{y}^2}{\overline{x}+\overline{y}}+\frac{\mu\overline{x}^2}{\overline{x}+\overline{y}}=\frac{\mu^2(\overline{x}\,\overline{y})}{(\overline{x}+\overline{y})^2} \quad \Rightarrow \quad (\overline{y}^2+\overline{x}^2)(\overline{x}+\overline{y})=\mu\overline{x}\,\overline{y}$ 5

Since $\overline{x},\overline{y}$ are variables, $(x+y)(x^2+y^2)=\mu xy$

09. The equations of two straight line L_1 and L_2 are $y-2x-5=0$ and $y-2x+1=0$ respectively. L_1 and L_2 are tangents to a circle. A third line L_3 is perpendicular to L_1 and L_2, and meets the circle at the point A and B; where $AB=\frac{6}{\sqrt{5}}$. Given that L_3 passes through the point $(2,2)$. Find the equation of the circle.



Let the gap between L_1 and L_2 is d.

Then $d=\frac{|1-(-5)|}{\sqrt{1^2+2^2}}=\frac{6}{\sqrt{5}}$. 5

$\therefore$ the radius $r=\frac{3}{\sqrt{5}}$.

But given that $AB=\frac{6}{\sqrt{5}}$.

$\therefore$ A and B are on the ends of a diameter. 5

Let its centre C lies on L_3. Then L_3 is

$y-2x-2=0$ 5

$\therefore$ let $C\equiv(t,2t+2)$

Let the line perpendicular to L_1 and L_2 is L_4. Then L_4 is

$$\frac{y-2}{x-2}=-\frac{1}{2}$$

$2y-4=-x+2$.

$x+2y-6=0$ 5

$$t+2(2t+2)-6=0$$

$$5t=2$$

$$t=\frac{2}{5}$$

$$\left(x-\frac{2}{5}\right)^2+\left(y-\frac{14}{5}\right)^2=\left(\frac{3}{\sqrt{5}}\right)^2 \quad 5$$

$$(5x-2)^2+(5y-14)^2=45$$

$$25x^2+25y^2-20x-140y+155=0$$

$$\underline{\underline{5x^2+5y^2-4x-28y+31=0}}$$

10. Solve the equation $\sin^{-1}\left(\frac{x}{x-1}\right)+2\tan^{-1}\left(\frac{1}{x+1}\right)=\frac{\pi}{2}$ where $x\in R-\{-1,1\}$.

$$\sin^{-1}\left(\frac{x}{x-1}\right)+2\tan^{-1}\left(\frac{1}{x+1}\right)=\frac{\pi}{2}$$

Let $\alpha+2\beta=\frac{\pi}{2};\alpha=\sin^{-1}\left(\frac{x}{x-1}\right),\beta=\tan^{-1}\left(\frac{1}{x+1}\right)$

$\sin\alpha=\frac{x}{x-1} \Rightarrow \tan\alpha=\frac{x}{\sqrt{1-2x}}$

$\tan\beta=\frac{1}{x+1}$

$\Rightarrow \alpha=\frac{\pi}{2}-2\beta \Rightarrow \tan\alpha=\cot 2\beta$ 5

$\Rightarrow \tan 2\beta\tan\alpha=1 \Rightarrow 2\tan\beta\tan\alpha=1-\tan^2\beta$ 5

5

$$2\frac{1}{1+x}\frac{x}{\sqrt{1-2x}}=1-\frac{1}{1+2x+x^2} \Rightarrow \frac{2x}{\sqrt{1-2x}}=\frac{x(2+x)}{1+x} \Rightarrow 2+2x=(2+x)\sqrt{1-2x}$$

$$\Rightarrow 4+8x+4x^2=(4+4x+x^2)(1-2x)$$

$$\Rightarrow 4x+3x^2=(4+4x+x^2)(-2x) \Rightarrow 4x+3x^2=(4+4x+x^2)(-2x)$$

$$\Rightarrow 12x+11x^2+2x^3=0 \Rightarrow x(2x^2+11x+12)=0 \Rightarrow x(2x+3)(x+4)=0$$

$$\Rightarrow x=0, \quad x=\frac{-3}{2}, \quad x=-4$$ 5

since, $x\in R-\{-1,1\}$ $\quad x=0$ and $x=-4$ 5

Part B

11. *a.* If α, β are the roots of the equation $ax^2+bx+c=0$ and γ, δ are the roots of the equation $lx^2+mx+n=0$, show that the equation whose roots are $\alpha\gamma+\beta\delta, \alpha\delta+\beta\gamma$ is $a^2l^2x^2-ablmx+nl(b^2-2ac)+ac(m^2-2nl)=0$.

b. $\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}, x\neq -p, x\neq -q$.

The roots of the above quadratic equation, where p, q and r are non zero constants, are equal in magnitude but opposite in sign. Show that the product of these roots is $-\frac{1}{2}(p^2+q^2)$.

c. If when the polynomial $f(x)$ is divided by x^2-x-6, the remainder is $7x-8$, then

i. find the remainder when $f(x)$ is divided by $x+2$.

ii. If $f(x)=2x^4+ax^3-20x^2-27x+b$, find the values of a and b. For these values of a and b, solve the equation $f(x)=7x-8$.

11. *a.* $ax^2+bx+c=0 \Rightarrow \alpha, \beta$ $\qquad$ $lx^2+mx+n=0 \Rightarrow \gamma, \delta$

$\alpha+\beta=-\frac{b}{a}, \quad \alpha\beta=\frac{c}{a}$ 5 $\qquad$ $\gamma+\delta=-\frac{m}{l}, \quad \gamma\delta=\frac{n}{l}$ 5

$$(\alpha\gamma+\beta\delta)+(\alpha\delta+\beta\gamma)=\alpha(\gamma+\delta)+\beta(\gamma+\delta)=(\gamma+\delta)(\alpha+\beta)=\frac{bm}{al}$$ 10

$$(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma)=\alpha^2\gamma\delta+\alpha\beta\gamma^2+\beta\alpha\delta^2+\beta^2\delta\gamma$$

$$=\gamma\delta(\alpha^2+\beta^2)+\alpha\beta(\gamma^2+\delta^2)$$

$$=\gamma\delta\{(\alpha+\beta)^2-2\alpha\beta\}+\alpha\beta\{(\gamma+\delta)-2\gamma\delta\}$$ 10

$$=\frac{n}{l}\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)+\frac{c}{a}\left(\frac{m^2}{l^2}-\frac{2n}{l}\right)$$ 5

$$=\frac{n}{la^2}(b^2-2ac)+\frac{c}{al^2}(m^2-2nl)$$ 5

The quadratic equation is

$$x^2-\frac{bm}{al}x+\frac{n}{la^2}\left(b^2-2ac\right)+\frac{c}{al^2}\left(m^2-2nl\right)=0 \quad 5$$

$$a^2l^2x^2-ablmx+nl\left(b^2-2ac\right)+ac\left(m^2-2nl\right)=0$$

45

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b. $$\frac{x+q+x+p}{(x+p)(x+q)}=\frac{1}{r}$$

$$\frac{2x+q+p}{(x+p)(x+q)}=\frac{1}{r} \quad 5$$

$$2rx+pr+qr=x^2+(p+q)x+pq$$

$$x^2+(p+q-2r)x+pq-pr-qr=0 \quad 5$$

$$x^2+(p+q-2r)x+pq-pr-qr=0 \Rightarrow \alpha,-\alpha$$

$$\alpha-\alpha=\frac{p+q-2r}{1} \quad 5$$

$$0=\frac{p+q-2r}{1} \quad 5$$

$$p+q-2r=0$$

$$p+q=2r \rightarrow (1) \quad 5$$

$$\alpha(-\alpha)=\frac{pq-pr-qr}{1} \quad 5$$

$$=pq-r(p+q)$$

$$=pq-\frac{(p+q)(p+q)}{2} \quad 5$$

$$=pq-\frac{(p+q)^2}{2}$$

$$=\frac{2pq-(p+q)^2}{2}$$

$$=\frac{2pq-p^2-q^2-2pq}{2} \quad 5$$

$$=-\frac{p^2+q^2}{2}$$

$$=-\frac{1}{2}\left(p^2+q^2\right)$$

40

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c. i. $f(x)=\left(x^2-x-6\right)q(x)+7x-8$ 10

$f(x)=(x-3)(x+2)q(x)+7x-8$ 5

When $x=-2$, $f(-2)=7\times(-2)-8=-22$ 5

$\therefore$ the remainder when $f(x)$ is divided by $x+2$ $=\underline{\underline{-22}}$ 5

Also when $x=3$, $f(3)=7\times(3)-8=13$ 5

ii. $f(x)=2x^4+ax^3-20x^2-27x+b$

$f(-2) = -22$ and $f(-2) = 2\times16 + a(-8) - 20\times4 - 27\times(-2\pi) + b$ 5

$-22 = 32 - 8a - 80 + 54 + b$ 5

$-8a + b = -28 \rightarrow (1)$ 5

$f(3) = 13$ and $f(3) = 2\times81 + a(27) - 20\times9 - 27\times3 + b$ 5

$13 = 162 + 27a - 180 - 81 + b$ 5

$27a + b = 112 \rightarrow (2)$ 5

$(2)-(1) \Rightarrow 35a = 140$

$\Rightarrow \quad \underline{\underline{a = 4}}$ 5

$(1) \Rightarrow -32 + b = -28$

$\Rightarrow \quad \underline{\underline{b = 4}}$ 5

65

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12. *a.* A cricket team consisting of 11 players is to be formed from 16 players of whom 4 can be bowlers and 2 can keep wicket and the rest can neither be bowler nor keep wicket. In how many different ways can a team be formed so that the teams contain

i. exactly 3 bowlers and 1 wicket keeper,

ii. at least 3 bowlers and at least 1 wicket keeper?

b. The nth term of the sequence $(2\times1!), (5\times2!), (10\times3!), (17\times4!), (26\times5!), ...$

satisfies a relationship of the form $U_r = \lambda(r+2)! + \mu(r+1)! + \gamma r!$.

Find the values of λ, μ and ν by taking $r = 1, 2$ and 3. Verify for $r = 4$.

Hence, prove that $\sum_{r=1}^{n} Ur = n(n+1)!$. Is the series $\sum_{r=1}^{n} Ur$ convergent? Verify your result.

12. *a.* *i.* Number of ways that 3 bowlers can be selected out of 4 $= {}^4C_3$ 5

Number of ways that 1 bowler can be selected out of 2 $= {}^2C_1$ 5

Number of ways that other 7 playas can be selected from the remaining 10 $= {}^{10}C_7$ 5

The total number of ways in which the cricket team can be formed $= {}^4C_3 \times {}^2C_1 \times {}^{10}C_7$ 5

$$= \frac{4!}{3!\times(4-3)!} \times \frac{2!}{1!\times(2-1)!} \times \frac{10!}{7!\times(10-7)!}$$

$$= 4\times2\times120$$

$$= \underline{\underline{960}}$$ 5

ii. Let bowlers as B, wicket keepers as W and players as P.

B	W	P	
3	1	7	${}^4C_3 \times {}^2C_1 \times {}^{10}C_7 = 4\times2\times120 = 960$ 5
3	2	6	${}^4C_3 \times {}^2C_2 \times {}^{10}C_6 = 4\times1\times210 = 840$ 5
4	1	6	${}^4C_4 \times {}^2C_1 \times {}^{10}C_6 = 1\times2\times210 = 420$ 5
4	2	5	${}^4C_4 \times {}^2C_2 \times {}^{10}C_5 = 1\times1\times252 = 252$ 5

Total number of ways $= \underline{\underline{2472}}$ 5

50

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b. $U_r = \lambda(r+2)! + \mu(r+1)! + \gamma r!$

when $r=1$, $U_1 = \lambda\times3! + \mu\times2! + \gamma\times1! \Rightarrow 6\lambda + 2\mu + \gamma = 2 \rightarrow (1)$ 5

when $r=2$, $U_2 = \lambda\times4! + \mu\times3! + \gamma\times2! \Rightarrow 12\lambda + 3\mu + \gamma = 5 \rightarrow (2)$ 5

when $r=3$, $U_3 = \lambda\times5! + \mu\times4! + \gamma\times3! \Rightarrow 20\lambda + 4\mu + \gamma = 10 \rightarrow (3)$ 5

$(2)-(1) \Rightarrow 6\lambda + \mu = 3 \rightarrow (4)$

$(3)-(2) \Rightarrow 8\lambda + \mu = 5 \rightarrow (5)$

$(5)-(4) \Rightarrow 2\lambda = 2$

$\underline{\underline{\lambda = 1}}$ 5

$(4) \Rightarrow 6 + \mu = 3$

$\underline{\underline{\mu = -3}}$ 5

$(1) \Rightarrow 6 - 6 + \gamma = 2$

$\underline{\underline{\gamma = 2}}$ 5

$U_r = (r+2)! - 3(r+1)! + 2r!$ 5

when $r=4$,

$RHS = 6! - 3\times5! + 2\times4!$ 5

$= 720 - 360 + 48$

$= 408$

$= 17\times24$

$= 17\times4!$ 5

$= \underline{\underline{LHS}}$

The result is true for $r=4$, 5

$U_r = (r+2)! - 3(r+1)! + 2r!$

when $r=1$, $\quad U_1 = 3! \;-\; 3\times 2! \;+\; 2\times 1!$

when $r=2$, $\quad U_2 = 4! \;-\; 3\times 3! \;+\; 2\times 2!$ 10

when $r=3$, $\quad U_3 = 5! \;-\; 3\times 4! \;+\; 2\times 3!$

when $r=4$, $\quad U_2 = 6! \;-\; 3\times 5! \;+\; 2\times 4!$

$\quad . \quad . \quad\quad . \quad\quad .$

$\quad . \quad . \quad\quad . \quad\quad .$

$\quad . \quad . \quad\quad . \quad\quad .$

when $r=n-2$, $\quad U_{n-2} = n! \;-\; 3(n-1)! \;+\; 2(n-2)!$

when $r=n-1$, $\quad U_{n-1} = (n+1)! \;-\; 3n! \;+\; 2(n-1)!$ 5

when $r=n$, $\quad U_n = (n+2)! \;-\; 3(n+1)! \;+\; 2n!$

$$\sum_{r=1}^{n} U_r = (n+2)! - 2(n+1)! - 2! + 2! \quad 5$$

$$= (n+1)!(n+2-2)$$

$$= \underline{\underline{n\times(n+1)!}} \quad 5$$

$$n \xrightarrow{\lim} \infty \quad \sum_{r=1}^{n} U_r = \xrightarrow{\lim} \infty \quad n\times(n+1)!$$

$$= \underline{\underline{\infty}} \quad 5$$

The series is not convergent. 5

13. *a.* If $A=\begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix}$, $B=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $C=\begin{pmatrix} 5 \\ 13 \end{pmatrix}$, find the values of a and b such that $AB=C$.

For above values of a and b, show that $A^2=5A+2I$.

The inverse of A is denoted by A^{-1} and I is the 2×2 identity matrix. Show further that

i. $A^3=27A+10I$ ii. $A^{-1}=\frac{1}{2}(A-5I)$

Hence, find A^{-1}

b. Consider the complex numbers $z=\frac{2+2i}{1-i}$ and $w=\frac{4}{1-\sqrt{3}i}$. Calculate the modulus of z and the modulus of w. Find the argument of z and the argument of w.

Plot the complex numbers $z=\frac{2+2i}{1-i}$ and $w=\frac{4}{1-\sqrt{3}i}$ in the Argand diagram. Find the position of $z+w$.

The points A, B and C represent the numbers $z, z+w$ and w respectively. The origin of the Argand diagram is denoted by O. By considering the quadrilateral $OABC$ and the argument of $z+w$, show that $\tan\frac{5\pi}{12}=2+\sqrt{3}$.

c. A complex number is defined as $z=\cos\theta+i\sin\theta; -\pi<\theta\le\pi$ and $n\in z^+$. Applying the De Movier's theorem, show that

i. $z^n+\frac{1}{z^n}=2\cos n\theta$

ii. $z^n-\frac{1}{z^n}=2i\sin n\theta$

iii. $8\sin^4\theta=\cos 4\theta-4\cos 2\theta+3$

Hence, solve the equation $8\sin^4\theta+5\cos 2\theta=3$; where $-\pi<\theta\le\pi$.

13. *a.* $A=\begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix}$, $B=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $C=\begin{pmatrix} 5 \\ 13 \end{pmatrix}$

$AB=C \Rightarrow \begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 13 \end{pmatrix} \Rightarrow 3a+2=5 \Rightarrow \underline{\underline{a=1}}$ and $9+b=13 \Rightarrow \underline{\underline{b=4}}$

5 5

Then $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$A^2 = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}7 & 10\\ 15 & 22\end{pmatrix} \quad 5$$

$$5A+2I = 5\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}+2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}5 & 10\\ 15 & 20\end{pmatrix}+\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}7 & 10\\ 15 & 22\end{pmatrix}$$

$5 \quad \Rightarrow A^2 = 5A+2I$

i.

$$A^2 = 5A+2I$$
$$A^3 = 5A^2+2IA \quad 5$$
$$A^3 = 5(5A+2I)+2A \quad 5$$
$$\underline{\underline{A^3 = 27A+10I}}$$

ii.

$$A^2 = 5A+2I$$
$$A^2A^{-1} = 5AA^{-1}+2IA^{-1} \quad 5$$
$$AI = 5I+2A^{-1} \quad 5$$
$$\underline{\underline{A^{-1} = \frac{1}{2}(A-5I)}}$$

$$A^{-1} = \frac{1}{2}\left(\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}-5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$
$$= \frac{1}{2}\left(\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}-\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}\right) \quad 5$$
$$= \frac{1}{2}\begin{pmatrix}-4 & 2\\ 3 & -1\end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix}-2 & 1\\ 3/2 & -1/2\end{pmatrix}}} \quad 5$$

50

---

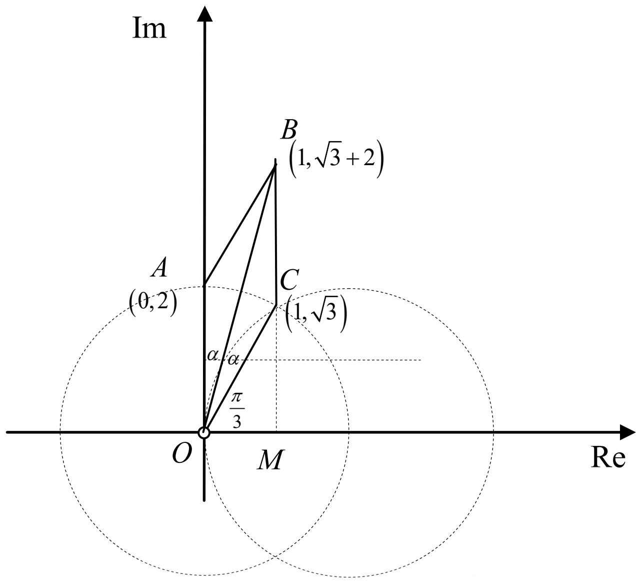
b. $z = \dfrac{2+2i}{1-i} = \dfrac{(2+2i)(1+i)}{1-i^2}$ $\qquad$ $w = \dfrac{4}{1-\sqrt{3}i} = \dfrac{4(1+\sqrt{3}i)}{1-3i^2} = \dfrac{(2+2i)(1+i)}{1-i^2}$ $\quad 5$

$= \dfrac{2+4i+2i^2}{1-i^2} = \dfrac{4i}{2} = 2i$ $\qquad$ $= \dfrac{4(1+\sqrt{3}i)}{4} = 1+\sqrt{3}i$ $\quad 5$

$\therefore |z| = \underline{\underline{2}}$ $\qquad$ $\therefore |w| = \underline{\underline{2}}$ $\quad 5$

$\arg(z) = \underline{\underline{0}}$ $\qquad$ $\arg(w) = \tan^{-1}\sqrt{3} = \underline{\underline{\dfrac{\pi}{3}}}$ $\quad 5$

$A \equiv z, \quad B \equiv z+w, \quad C \equiv w$



15

$OA = AB = BC = CO = 2$

$\therefore OABC$ is a rhombus. 5

$\therefore B\hat{O}C = \frac{1}{2} A\hat{O}C = \frac{\pi}{12}$

$\therefore Arg(z+w) = \frac{5\pi}{12}$ 5

$\therefore \tan\frac{5\pi}{12} = \frac{BM}{OM} = \frac{2+\sqrt{3}}{1} = 2+\sqrt{3}$

45

c. $z = \cos\theta + i\sin\theta \,; -\pi < \theta \le \pi$

$z^n = (\cos\theta + i\sin\theta)^n$

$z^n = \cos n\theta + i\sin n\theta \quad \rightarrow \quad (1)$ 5

$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta}$

$\frac{1}{z} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta}$

$\frac{1}{z} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$

$\frac{1}{z} = \cos\theta - i\sin\theta$ 5

$\left(\frac{1}{z}\right)^n = (\cos\theta - i\sin\theta)^n$

$\frac{1^n}{z^n} = \left[\cos(-\theta) + i\sin(-\theta)\right]^n$

$\frac{1}{z^n} = \cos(-n\theta) + i\sin(-n\theta)$

$\frac{1}{z^n} = \cos n\theta - i\sin n\theta \quad \rightarrow \quad (2)$ 5

$(1)+(2),\quad \underline{\underline{z^n+\frac{1}{z^n}=2\cos n\theta}}$

5

$(1)-(2),\quad \underline{\underline{z^n-\frac{1}{z^n}=2i\sin n\theta}}$

$z^n-\frac{1}{z^n}=2i\sin n\theta$

For $n=1$, $z-\frac{1}{z}=2i\sin\theta$ 5

$\left(z-\frac{1}{z}\right)^4=2^4 i^4\sin^4\theta$

$z^4-4z^3.\frac{1}{z}+6z^2.\frac{1}{z^2}-4z.\frac{1}{z^3}+\frac{1}{z^4}=16.1.\sin^4\theta$ 5

$16\sin^4\theta=z^4+\frac{1}{z^4}-4\left(z^2+\frac{1}{z^2}\right)+6$

When $n=4$,

$z^4+\frac{1}{z^4}=2\cos 4\theta$

5

When $n=2$,

$z^2+\frac{1}{z^2}=2\cos 2\theta$

$16\sin^4\theta=2\cos 4\theta-4\cos 2\theta+6$

$\underline{\underline{8\sin^4\theta=\cos 4\theta-2\cos 2\theta+3}}$ 5

$8\sin^4\theta+5\cos 2\theta=3$; where $-\pi<\theta\le\pi$.

$8\sin^4\theta+5\cos 2\theta=\cos 4\theta+3\cos 2\theta+3=3$

$\cos 4\theta+3\cos 2\theta=0$

$2\cos^2 2\theta-1+3\cos 2\theta=0$

$2\cos^2 2\theta+3\cos 2\theta-1=0$

$(2\cos 2\theta-1)(\cos 2\theta+1)=0$ 5

$(2\cos 2\theta-1)=0$ or $(\cos 2\theta+1)=0$

$\cos 2\theta=\frac{1}{2}$ or $\cos 2\theta=-1$

$\cos 2\theta=\cos\frac{\pi}{3}$ or $\cos 2\theta=\cos\pi$

$2\theta=2n\pi\pm\frac{\pi}{3}$ or $2\theta=2n\pi\pm\pi$

$\theta=n\pi\pm\frac{\pi}{6}$ or $\theta=n\pi\pm\frac{\pi}{2}$ 5

$\theta=\left\{\frac{\pi}{6},-\frac{\pi}{6},\frac{\pi}{2},-\frac{\pi}{2},\frac{5\pi}{6},-\frac{5\pi}{6}\right\}$ 5

55

14. *a.* Let $f(x)=\dfrac{-2x}{(x-1)^2}$ for $x\neq 1$.

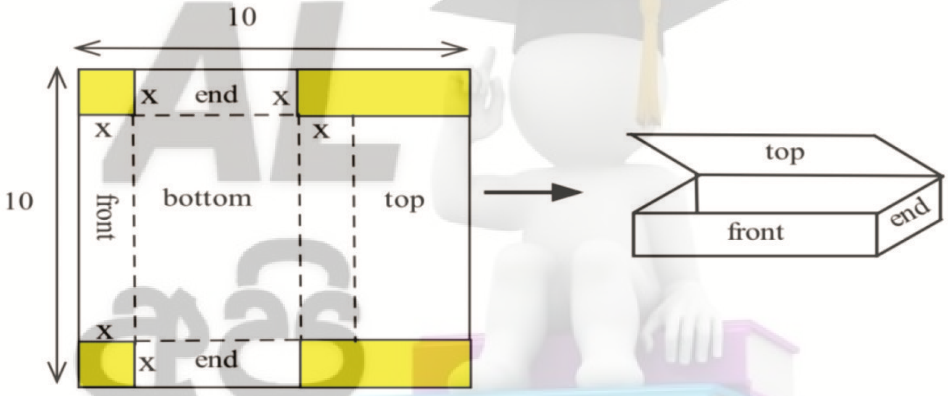
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{2(x+1)}{(x-1)^3}$ for $x\neq 1$.

It is given that $f''(x)=-\dfrac{4(x+2)}{(x-1)^4}$ for $x\neq 1$.

Find the x-coordinates of the points of inflection of the graph of $y=f(x)$.

Sketch the graph of for $y=f(x)$ indicating the asymptotes, y-intercept and the turning points.

b. The figure shows a net made using a square shaped piece of cardboard which the side length is 10 cm for a box.



If the volume of the box formed is V, show that $V=2\left(25x-10x^2+x^3\right)$.

Find the dimensions of the box having the maximum volume.

---

14. *a.*

$$f(x)=\frac{-2x}{(x-1)^2}\ ;\ x\neq 1$$

$$f'(x)=\frac{(x-1)^2\times(-2)-(-2x)\times 2(x-1)}{(x-1)^4} \quad 10$$

$$f'(x)=\frac{(x-1)\times(-2)+(2x)\times 2}{(x-1)^3}$$

$$f'(x)=\frac{-2x+2+4x}{(x-1)^3}$$

$$f'(x)=\frac{2x+2}{(x-1)^3}$$

$$f'(x)=\frac{2(x+1)}{(x-1)^3} \quad 5$$

15

$$f''(x)=-\frac{4(x+2)}{(x-1)^4}$$

For the points of inflection, $f''(x)=0$. 5

Then $x=-2$

Range of x	$(-\infty,-2)$	$(-2,\infty)$
Sign of $f''(x)$,	$(+)$	$(-)$
While x is increasing,	Concave up	Concave down

10

The point of inflection $\equiv\left(-2,\dfrac{4}{9}\right)$ 5

For the stationary points, $f'(x)=\dfrac{2(x+1)}{(x-1)^3}=0$

Then $x=-1 \Rightarrow \left(-1,\dfrac{1}{2}\right)$ is a stationary point. 5

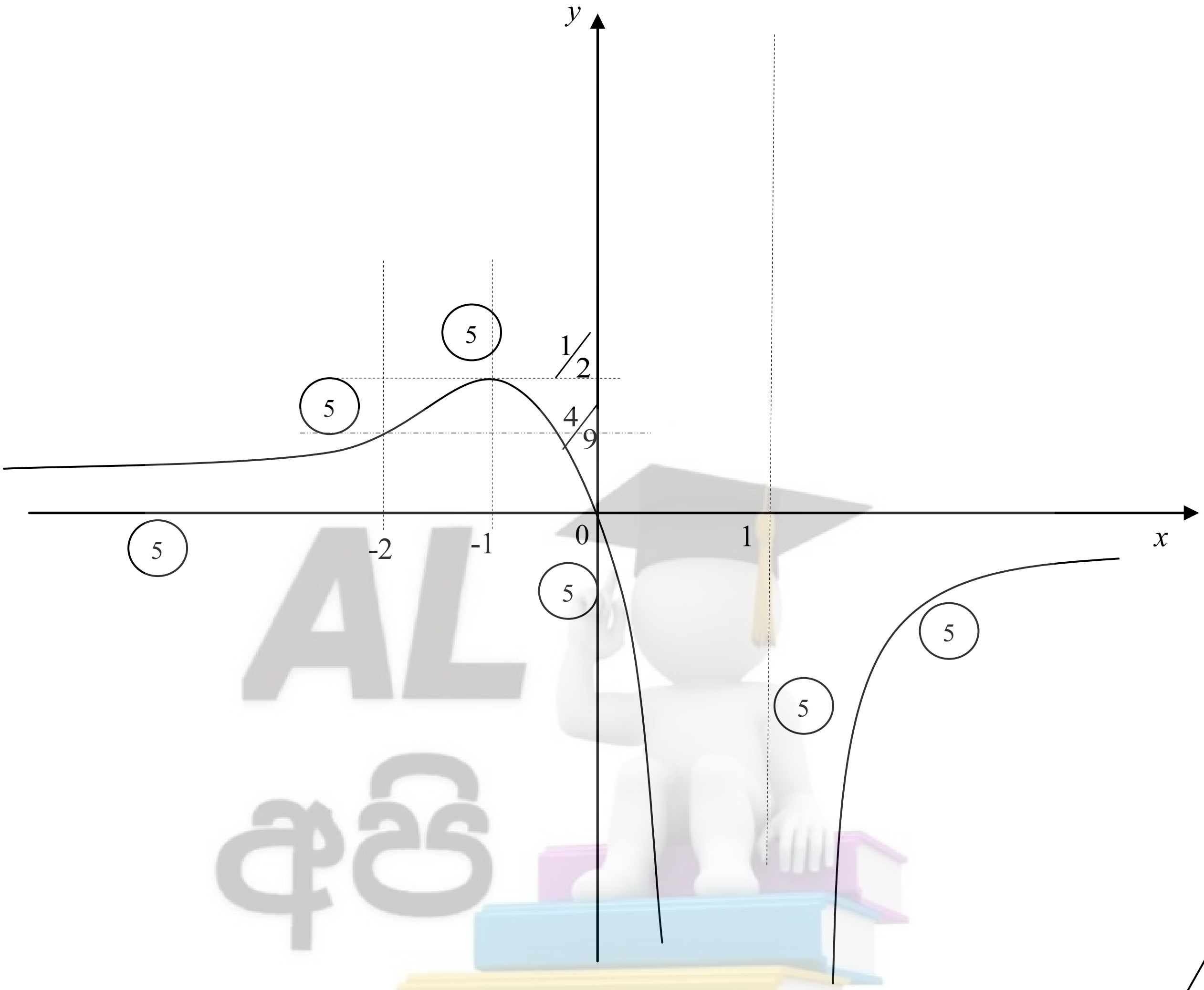
$x=1$ is a vertical asymptote.

$y=0$ is a horizontal asymptote.

The intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing

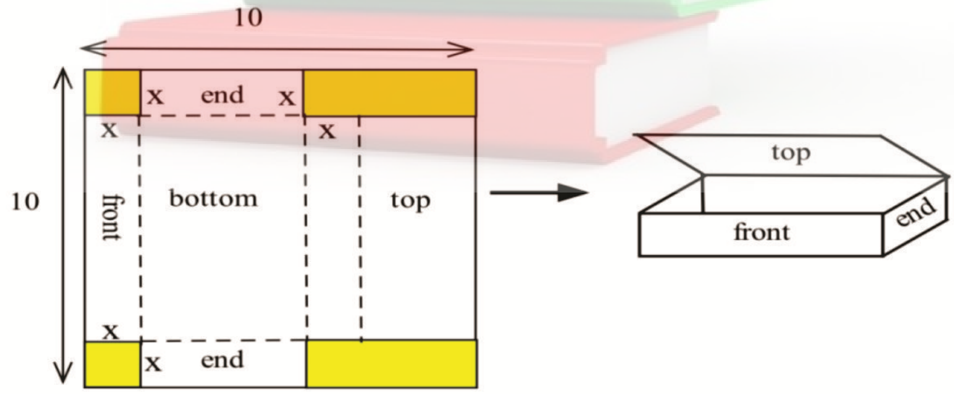
Range of x	$(-\infty,-1)$	$(-1,1)$	$(1,-\infty)$
Sign of $f'(x)$	$(+)$	$(-)$	$(+)$
While x is increasing,	$f(x)$ is increasing	$f(x)$ is decreasing	$f(x)$ is increasing
	5	5	5

$\Rightarrow \left(-1,\dfrac{1}{2}\right)$ is a local maximum point. 5


y
x
0
1
-1
-2
1/2
4/9
5
5
5
5
5
5
5

75

b.


10
10
x
end
x
x
x
front
bottom
top
x
x
end
top
front
end

$$V=(10-2x)\frac{(10-2x)}{2}x$$ 10

$$V=2x(5-x)^2$$

$$V=2x\left(x^2-10x+25\right)$$

$$V=2\left(x^3-10x^2+25x\right)$$

$$\frac{d}{dx}V=2\left(3x^2-20x+25\right)$$ 10

$$\frac{d}{dx}V=2(3x-5)(x-5)$$

$\frac{d}{dx}V=0$ for maximum or minimum volume.

Then $x=\frac{5}{3}$ or $x=5$ 5

Being $x=5$ is impossible.

$\therefore \quad \underline{\underline{x=\frac{5}{3}\,cm}}$ 5

Range of x	$\left(0,\frac{5}{3}\right)$	$\left(\frac{5}{3},5\right)$
Sign of $\frac{d}{dx}V$	$(+)$	$(-)$
While x is increasing,	V is increasing	V is decreasing

10

$\therefore \quad V$ is maximum for $x=\frac{5}{3}\,cm$ 5

$\therefore \quad$ length $=\frac{20}{3}\,cm$

$\therefore \quad$ breadth $=\frac{10}{3}\,cm$ 15

$\therefore \quad$ height $=\frac{5}{3}\,cm$

60

15. *a*. Find the value of the constants A, B and C such that

$$\frac{4}{(1-u)^2(1+u)} = \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}.$$ Hence, integrate $\frac{4}{(1-u)^2(1+u)}$ with respect to u.

Hence, by using a suitable substitution show that $\int_0^{\frac{\pi}{6}} \frac{4}{\cos x(1-\sin x)}dx = 2 + \ln 3$.

b. Using a suitable substitution and the method of **integration by parts,** find $\int e^{\sin^{-1}x}dx$.

c. Find $\frac{d}{dx}(2^x)$. When c is a constant, evaluate $\int_{-c}^{c} \frac{2^x}{2^x+1}dx$.

When $0 \le c \le \pi/2$, if $I = \int_{-c}^{c} \frac{\cos x}{1+2^x}dx$ and $J = \int_{-c}^{c} \frac{a^x \cos x}{1+2^x}dx$,

i. By using the substitution $x = -t$ or otherwise, show that $I = J$

ii. Obtain $I + J$.

Hence, find the value of J when $c = \pi/6$.

---

15. *a*. $\frac{4}{(1-u)^2(1+u)} = \frac{A}{(1-u)^2} + \frac{B}{1-u} + \frac{C}{1+u}$.

$4 = A(1+u) + B(1-u)(1+u) + C(1-u)^2$

$4 = A(1+u) + B(1-u^2) + C(1-2u+u^2)$

$\underline{u = -1}$

$4 = C(4)$

$\underline{\underline{C = 1}}$

$\underline{u = 1}$

$4 = 2A$

$\underline{\underline{A = 2}}$

$\underline{u^2}$

$0 = -B + C$

$B = C$

$\underline{\underline{B = 1}}$

10

$$\frac{4}{(1-u)^2(1+u)}=\frac{2}{(1-u)^2}+\frac{1}{1-u}+\frac{1}{1+u}$$

$$\int\frac{4}{(1-u)^2(1+u)}du=2\int\frac{1}{(1-u)^2}du+\int\frac{1}{1-u}du+\int\frac{1}{1+u}du$$

$$\int\frac{4}{(1-u)^2(1+u)}du=-2\int(1-u)^{-2}(-1)du-\int\frac{-1}{1-u}du+\int\frac{1}{1+u}du \quad 5$$

5 5

$$\int\frac{4}{(1-u)^2(1+u)}du=-2\frac{(1-u)^{-2+1}}{-1}-\ln|1-u|+\ln|1+u|$$

$$\int\frac{4}{(1-u)^2(1+u)}du=\frac{2}{1-u}+\ln|1+u|-\ln|1-u|$$

$$\int\frac{4}{(1-u)^2(1+u)}du=\underline{\underline{\frac{2}{1-u}+\ln\left|\frac{1+u}{1-u}\right|+C}} \quad 5$$

20

$$\int_0^{\frac{\pi}{6}}\frac{4}{\cos x(1-\sin x)}dx.$$

Let $\sin x=u$ 5

$$\frac{du}{dx}=\cos x$$

When $x=0$, then $u=\sin 0 \quad \Rightarrow u=0$

5

When $x=\frac{\pi}{6}$, then $u=\sin\frac{\pi}{6} \quad \Rightarrow u=\frac{1}{2}$

$$\int_0^{\frac{\pi}{6}}\frac{4}{\cos x(1-\sin x)}dx=\int_0^{\frac{\pi}{6}}\frac{4\cos x}{\cos^2 x(1-\sin x)}dx \quad 5$$

$$\int_0^{\frac{\pi}{6}}\frac{4}{\cos x(1-\sin x)}dx=\int_0^{\frac{\pi}{6}}\frac{4\cos x}{(1-\sin^2 x)(1-\sin x)}dx$$

$$\int_0^{\frac{\pi}{6}}\frac{4}{\cos x(1-\sin x)}dx=\int_0^{\frac{1}{2}}\frac{4}{(1-u^2)(1-u)}du \quad 5$$

$$=\left[\frac{2}{1-u}+\ln\left|\frac{1+u}{1-u}\right|\right]_0 \quad 5$$

$$=\left[\frac{2}{1-\frac{1}{2}}+\ln\left|\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right|\right]-\left[\frac{2}{1}+\ln|1|\right]$$

$$=\left[\frac{4}{1}+\ln\left|\frac{3}{1}\right|\right]-[2+0]$$

$$=\underline{\underline{2+\ln|3|}} \quad 5$$

30

---

b. $\int e^{\sin^{-1}x}dx$

Let $\sin^{-1}x = u$. Then $x = \sin u \Rightarrow \dfrac{dx}{du} = \cos u$ 5 5

$\therefore \int e^{\sin^{-1}x}dx = \int e^u \cos u du \rightarrow (A)$

$$\int e^u \cos u du = \int \cos u \frac{de^u}{du} du$$

$$\int e^u \cos u du = e^u \cos u + \int e^u \sin u du \quad 5$$

$$\int e^u \cos u du = e^u \cos u + \left[e^u \sin u - \int e^u \cos u du\right] \quad 5$$

$$\int e^u \cos u du = e^u (\cos u + \sin u) - \int e^u \cos u du$$

$$2\int e^u \cos u du = e^u (\cos u + \sin u)$$

$$\int e^u \cos u du = \frac{e^u}{2}(\cos u + \sin u) + C \quad 5$$

$$\therefore (A) \Rightarrow \int e^{\sin^{-1}x}dx = \frac{e^u}{2}(\cos u + \sin u) + C$$

$$\therefore \int e^{\sin^{-1}x}dx = \frac{e^{\sin^{-1}x}}{2}\left(x + \sqrt{1-x^2}\right) + C \quad 5$$

30

---

c. Let $u = 2^x$. Then $\ln u = x \ln 2$

$$\frac{1}{u}\frac{du}{dx} = \ln 2$$

$$\therefore \frac{du}{dx} = u \ln 2$$

$$\therefore \frac{d\left(2^x\right)}{dx} = 2^x \ln 2 \quad 5$$

5

$$\int_{-c}^{c} \frac{2^x}{1+2^x}dx = \frac{1}{\ln 2}\int_{-c}^{c} \frac{2^x \ln 2}{1+2^x}dx \quad 5$$

$$\int_{-c}^{c} \frac{2^x}{1+2^x}dx = \frac{-1}{\ln 2}\left[-\ln\left|1+2^x\right|\right]_{-c}^{c} \quad 5$$

$$\int_{-c}^{c} \frac{2^x}{1+2^x}dx = \frac{-1}{\ln 2}\left[-\ln\left|1+2^x\right|\right]_{-c}^{c}$$

$$\int_{-c}^{c} \frac{2^x}{1+2^x}dx = \frac{1}{\ln 2}\left[\ln\left|1+2^c\right| - \ln\left|1+2^{-c}\right|\right]_{-c}^{c} \quad 5$$

$$\int_{-c}^{c} \frac{2^x}{1+2^x}dx = \frac{1}{\ln 2}\ln\left|\frac{1+2^c}{1+2^{-c}}\right| \quad 5$$

20

When $0 \le c \le \pi/2$ $\quad I = \int_{-c}^{c} \frac{\cos x}{1+2^x} dx \quad J = \int_{-c}^{c} \frac{a^x \cos x}{1+2^x} dx$

$$I = \int_{-c}^{c} \frac{\cos x}{1+2^x} dx$$

$x = -t$

When $x = -c$, then $t = c$

When $x = c$, then $t = -c$

$\frac{dx}{dt} = -1$ 5

$$I = \int_{-c}^{c} \frac{\cos x}{1+2^x} dx$$

$$I = \int_{c}^{-c} \frac{\cos(-t)}{1+2^{-t}} (-dt) \quad 5$$

$$I = \int_{-c}^{c} \frac{\cos t}{1+ 1/2^t} dt$$

$$I = \int_{-c}^{c} \frac{2^t \cos t}{2^t + 1} dt$$

$$I = \int_{-c}^{c} \frac{2^x \cos x}{2^x + 1} dx \quad 5$$

$I = J \rightarrow (1)$

$$I + J = \int_{-c}^{c} \frac{\cos x}{1+2^x} dx + \int_{-c}^{c} \frac{2^x \cos x}{2^x + 1} dx$$

$$I + J = \int_{-c}^{c} \cos x dx \quad 5$$

$I + J = [\sin x]_{-c}^{c}$

$I + J = \sin c - \sin(-c)$

$I + J = \sin c + \sin c$

$I + J = 2\sin c \rightarrow (2)$ 5

(1) and (2), $\quad 2J = 2\sin c$

$J = \sin c$ 5

When $c = \frac{\pi}{6}$, $\quad J = \sin\frac{\pi}{6}$

$\underline{\underline{J = \frac{1}{2}}}$ 5

35

16. Show that the perpendicular distance from the point (h,k) to the line $ax+by+c=0$ is given by $\left|\dfrac{ax+bx+c}{\sqrt{a^2+b^2}}\right|$.

The points A and B are on the circle $(x+4)^2+(y+2)^2=25$ whose centre is C. The tangents drawn to the circle at A and B meet at the point $D(3,-3)$.

i. Find the equations of the tangents drawn at A and B.

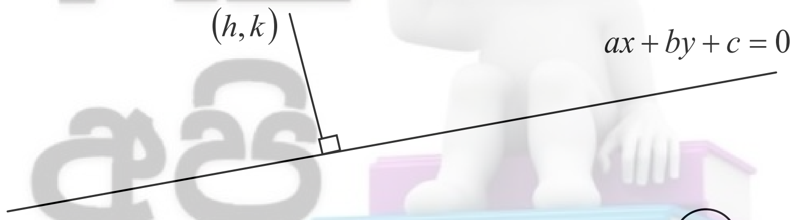
ii. Find the coordinates of the points A and B.

iii. Find the angle $A\hat{D}B$.

iv. Find the maximum and minimum distances of the point D from the circle.

v. Find the areas of the quadrilateral $ADBC$ and the triangle ADB.

vi. Find the equation of the circle circumscribing the triangle ADB. Also find the intercepts made by this circle on the coordinate axes.



5

Gradient of line is $-\dfrac{a}{b}$ and that of perpendicular is $\dfrac{b}{a}$. Let any point on the perpendicular is (x,y).

Then $\dfrac{y-k}{x-h}=\dfrac{b}{a} \Rightarrow \dfrac{x-h}{a}=\dfrac{y-k}{b}=t;$ t is a parameter. 5

Then $x=at+h,\quad y=bt+k$, $(at+h,bt+k)$ is any point on the perpendicular. 5

When $(at+h,bt+k)$ is on the line,

$a(at+h)+b(bt+y)+c=0 \Rightarrow (a^2+b^2)t+ah+by+c=0$ 5

$$\Rightarrow t=-\frac{(ah+by+c)}{(a^2+b^2)}$$

Now the perpendicular distance to the line from the point is $\sqrt{(at+h-h)^2+(bt+k-k)^2}=\sqrt{(a^2+b^2)t^2}$ 5

$$= \sqrt{\left(a^2+b^2\right)\left[-\frac{\left(ah+by+c\right)}{\left(a^2+b^2\right)}\right]^2}$$

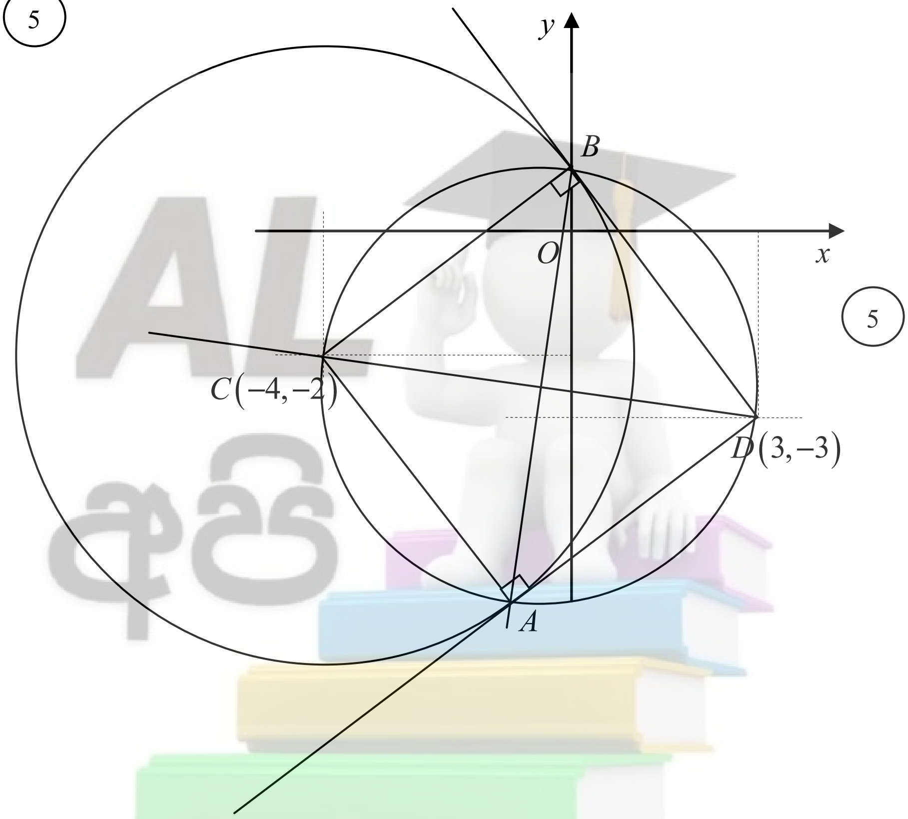
$$= \frac{\left|ah+by+c\right|}{\sqrt{a^2+b^2}} \quad 5$$

30

$\left(x+4\right)^2+\left(y+2\right)^2=25$

$\Rightarrow \quad C \equiv \left(-4,-2\right), \quad r=5$

5



5

Let any line passes through $D\left(3,-3\right)$ is $\dfrac{y+3}{x-3}=m$. i.e. $mx-y-3\left(m+1\right)=0$. 5

Considering the perpendicular distance from $C\left(-4,-2\right)$ to

$$5=\frac{\left|m\times\left(-4\right)-\left(-2\right)-3\left(m+1\right)\right|}{\sqrt{m^2+\left(-1\right)^2}} \quad 5$$

$$5\sqrt{m^2+1}=\left|-4m+2-3\left(m+1\right)\right|$$

$$25\left(m^2+1\right)=49m^2+14m+1$$

$$0=24m^2+14m-24$$

$$0=12m^2+7m-12$$

$$0=\left(4m-3\right)\left(3m+4\right)$$

$$m=\frac{3}{4} \quad or \quad m=\frac{-4}{3} \quad 5$$

Equation of AD

$$\frac{3}{4}x - y - 3\left(\frac{3}{4}+1\right) = 0$$

$$\underline{\underline{3x - 4y - 21 = 0}} \quad 5$$

Equation of BD

$$\left(\frac{-4}{3}\right)x - y - 3\left(\left(\frac{-4}{3}\right)+1\right) = 0$$

$$\underline{\underline{-4x - 3y + 3 = 0}} \quad 5$$

Equation of BC

$$\frac{y+2}{x+4} = \frac{3}{4} \quad 5$$

$$4y + 8 = 3x + 12$$

$$3x - 4y + 4 = 0$$

Equation of AC

$$\frac{y+2}{x+4} = -\frac{4}{3} \quad 5$$

$$3y + 6 = -4x - 16$$

$$4x + 3y + 22 = 0$$

Coordinates of A

$$\frac{3x-21}{4} = \frac{-4x-22}{3} \quad 5$$

$$9x - 63 = -16x - 88$$

$$25x = -25$$

$$x = -1$$

$$y = \frac{-24}{4}$$

$$y = -6$$

$$\underline{\underline{A \equiv (-1, -6)}} \quad 5$$

Coordinates of B

$$\frac{-4x+3}{3} = \frac{3x+4}{4} \quad 5$$

$$-16x + 12 = 9x + 12$$

$$x = 0, y = 1$$

$$\underline{\underline{B \equiv (0, 1)}} \quad 5$$

$$A\hat{D}B = \tan^{-1}\left|\frac{\frac{3}{4}-\frac{-4}{3}}{1+\frac{3}{4}\times\frac{-4}{3}}\right| \quad 5$$

$$A\hat{D}B = \frac{\pi}{2} \quad 5$$

$$CD = \sqrt{(-4-3)^2+(-2+3)^2} \quad 5$$

$$CD = 5\sqrt{2}$$

the maximum distance of the point D from the circle $= \underline{\underline{5\sqrt{2}+5}}$

5

the minimum distance of the point D from the circle $= \underline{\underline{5\sqrt{2}-5}}$

the area of the quadrilateral $ADBC$ $= \underline{\underline{25}}$ 5

the area of the triangle $ADB = \underline{\underline{12.5}}$ 5

the equation of the circle circumscribing the triangle ADB

radius $= \frac{5\sqrt{2}}{2}$, 5 Centre $\equiv \left(\frac{-4+3}{2}, \frac{-2-3}{2}\right)$

$$\equiv \left(\frac{-1}{2}, \frac{-5}{2}\right) \quad 5$$

$$\left(x+\frac{1}{2}\right)^2+\left(y+\frac{5}{2}\right)^2 = \frac{50}{4}$$

$$(2x+1)^2+(2y+5)^2 = 50$$

$$4x^2+4x+1+4y^2+20y+25 = 50$$

$$\underline{\underline{4x^2+4y^2+4x+20y-24=0}} \quad 5$$

the intercepts made by this circle on the coordinate axes.

For y intercepts, $x=0$. Then $4y^2+20y-24=0$

$$\Rightarrow y^2+5y-6=0$$

$$(y+6)(y-1)=0$$

$$y=-6, y=1 \quad 5$$

For x intercepts, $y=0$. Then $4x^2+4x-24=0$

$$x^2+x-6=0$$

$$(x+3)(x-2)=0$$

$$x=-3, x=2 \quad 5$$

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17. *a*. If the roots of the quadratic equation $x^2 + mx + n = 0$ are $\tan 30^\circ$ and $\tan 15^\circ$, then find the value of $2 + n - m$.

b. State and prove the ***sine rule*** for a any triangle ABC, in the usual notation.

In the triangle ABC, angles A, B and C are in an arithmetic progression. Show that $B = 60^\circ$ and $a^2 - ac + c^2 = b^2$.

Using the above results, prove that $\cos\left(\dfrac{A-C}{2}\right) = \dfrac{a+c}{2\sqrt{a^2 - ac + c^2}}$.

c. If $0 \le x \le \pi$, then solve the equation $16^{\sin^2 x} + 16^{\cos^2 x} = 10$.

17. *a*.

$x^2 + mx + n = 0 \quad \Rightarrow \tan 30^\circ, \tan 15^\circ$

$\tan 30^\circ + \tan 15^\circ = -m$ 5 5

$\tan 30^\circ \tan 15^\circ = n$

$\tan 45^\circ = \tan\left(30^\circ + 15^\circ\right)$ 5

$= \dfrac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ}$ 5

$1 = \dfrac{-m}{1-n}$ 5

$1 = n - m$

$2 + n - m = 2 + 1$

$= \underline{\underline{3}}$ 5

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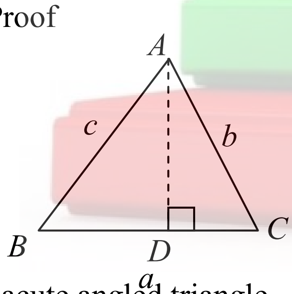
b. The ***sine rule*** for a triangle ABC, in the usual notation is $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. 5

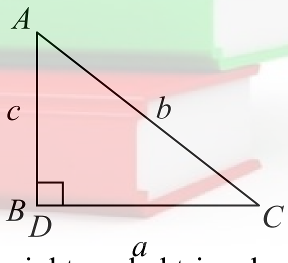
Proof



For acute angled triangle

$AD = c \sin B$

$AD = b \sin C$



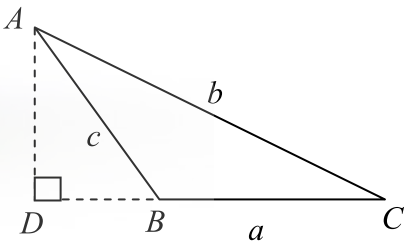
For right angled triangle

$AD = c$

$AD = c \sin 90^\circ$ 5

$AD = c \sin B$

$AD = b \sin C$



For obtuse angled triangle

$AD = c \sin\left(180^\circ - B\right)$

$AD = c \sin B$ 5

$AD = b \sin C$

$\Rightarrow \quad c \sin B = b \sin C \quad \Rightarrow \quad \dfrac{c}{\sin C} = \dfrac{b}{\sin B}$

Similarly, we can prove that $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ 5

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Angles A, B, C are in AP

Then

$$B - A = C - B$$

$$B = \frac{A+C}{2}$$

$$A + C = 2B \quad 5$$

But

$$A + B + C = \pi$$

$$B + 2B = \pi$$

$$\underline{\underline{B = \frac{\pi}{3}}} \quad 5$$

$$\cos B = \cos\frac{\pi}{3} \quad 5$$

$$5 \quad \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2}$$

$$a^2 + c^2 - b^2 = ac$$

$$\underline{\underline{a^2 - ac + c^2 = b^2}}$$

$$\frac{a+c}{2\sqrt{a^2 - ac + c^2}} = \frac{a+c}{2\sqrt{b^2}} \quad 5$$

$$= \frac{k\sin A + k\sin C}{2k\sin B} \quad 5$$

$$= \frac{\sin A + \sin C}{2\sin B}$$

$$= \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{2\sin B} \quad 5$$

$$= \frac{2\cos\left(\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right)}{4\sin\frac{B}{2}\cos\frac{B}{2}} \quad 5 \quad 5$$

$$= \frac{\cos\left(\frac{A-C}{2}\right)}{2\sin\frac{B}{2}}$$

$$= \frac{\cos\left(\frac{A-C}{2}\right)}{2\sin 30^\circ} \quad 5$$

$$= \underline{\underline{\cos\left(\frac{A-C}{2}\right)}} \quad 5$$

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b.

$$\left(2^4\right)^{\sin^2 x} + \left(2^4\right)^{\cos^2 x} = 10 \quad 5$$

$$\Rightarrow \left(2^4\right)^{\sin^2 x} + \left(2^4\right)^{1-\sin^2 x} = 10 \quad 5$$

$$\Rightarrow \left(2^4\right)^{\sin^2 x} + \frac{\left(2^4\right)}{\left(2^4\right)^{\sin^2 x}} = 10$$

$$\Rightarrow y^2 + \left(2^4\right) = 10y\ ; \quad y = \left(2^4\right)^{\sin^2 x}$$

$$\Rightarrow y^2 - 10y + 16 = 0 \quad 5$$

$$\Rightarrow (y-8)(y-2) = 0$$

$$\Rightarrow \left(2^4\right)^{\sin^2 x} = 8 \quad 5 \qquad \left(2^4\right)^{\sin^2 x} = 2 \quad 5$$

$$\Rightarrow 2^{4\sin^2 x} = 2^3 \qquad 2^{4\sin^2 x} = 2^1$$

$$\Rightarrow \sin^2 x = \frac{3}{4} \quad 5 \qquad \sin^2 x = \frac{1}{4} \quad 5$$

$$\Rightarrow \sin x = \pm\frac{\sqrt{3}}{2} \qquad \sin x = \pm\frac{1}{2}$$

Given that x is between 0 and $\frac{\pi}{2}$,

$$\underline{\underline{x = \frac{\pi}{3}}} \quad 5 \qquad \underline{\underline{x = \frac{\pi}{6}}} \quad 5$$

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